Linear Independence
Reading: Lay 1.7

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In this section, we discuss the concept of linear dependence and independence. I am going to introduce the definitions and then work some examples and try to "flesh out" the meaning and implications of the definition.

These definitions might seem strange at first, and you might not understand why we care about dependence or independence (Lay does not do a good job explaining why). So I have included an appendix to these notes, Appendix A, which describes one reason why we care about these things.

1 Definitions of dependence and independence

Definition 1.1. An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is called linearly independent if the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = x_2 = \ldots = x_p = 0$.

Definition 1.2. An indexed set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called linearly dependent if the vector equation

$$x_1\mathbf{u}_1 + \ldots + x_p\mathbf{u}_p = \mathbf{0}$$

holds for some weights $x_1, x_2, \ldots, x_p$ which are not all zero.

The first thing to note is that a given set of vectors is either linearly independent or linearly independent; it cannot be both or neither.

Example 1.3. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent or linearly independent?

The way to check is to look at the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}. \tag{1}$$

If this equation has a solution other than $x_1 = x_2 = x_3 = 0$, then they are linearly dependent; otherwise, they are linearly independent. So write the augmented matrix for (1):

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}. \tag{2}$$

We start on the computation of the RREF in the usual way. Add -1 times the top row to the second row:

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}. \tag{3}$$

We can actually answer the question just by looking at the matrix (3); we do not need to finish computing the RREF. Since there are only two nonzero rows in (3), there can only be at most two pivot positions in the RREF. But there are three variables x_1, x_2, x_3, so this means that x_3 is a free variable. This implies that there is a nontrivial solution, so the vectors are linearly independent.

The last paragraph was somewhat "advanced", so in case you did not follow, we will finish computing the RREF. Take the matrix (3) and add -1 times the first row to the fourth:

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -7 & 0 \end{bmatrix}.$$

This is the RREF of (2). It corresponds to the system of equations

$$\begin{aligned} x_1 + 6x_3 &= 0 \\ x_2 - 7x_3 &= 0. \end{aligned}$$

There is a free variable x_3, so there exists a solution to (1) which is not the trivial solution (for instance, take $x_1 = -6$, $x_2 = 7$, $x_3 = 1$). Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Example 1.4. Is the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly independent or dependent, if

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad ?$$

As before, we write the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}. \tag{4}$$

The first pivot position is the upper-left entry, and the column below it is already all zeros. So we move to the second pivot position and begin emptying the column underneath:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we have only two pivots, so there is a nontrivial solution. Thus $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent.

Example 1.5. What about the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent? Write the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Computing the RREF gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

There are no free variables, so the only solution to the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$$

is $x_1 = x_2 = 0$. Thus, the system is linearly independent.

1.1 Homogeneous equations

One small point to note is the connection of linear independence to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Suppose the matrix $A = [\mathbf{a}_1 \ldots, \mathbf{a}_n]$. Then the solutions of $A\mathbf{x} = \mathbf{0}$ are those vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ such that

$$x_1\mathbf{a}_1 + \ldots + \mathbf{a}_n = \mathbf{0}.$$

So the homogeneous equation has a nontrivial solution–that is, a solution $\mathbf{x} \neq \mathbf{0}$– if and only if the columns of A are linearly dependent.

2 Sets of one or two vectors

We will talk now about what linear dependence and independence "look like".

2.1 One vector

First, we consider sets of one vector, like $\{\mathbf{v}\}$. Such a set is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$. This is because the equation

$$x_1\mathbf{v}_1 = \mathbf{0}$$

is equivalent to the linear system

$$\begin{aligned} x_1 v_1 &= 0 \\ x_1 v_2 &= 0 \\ &\vdots \\ x_1 v_n &= 0. \end{aligned} \tag{5}$$

(here we use the notation $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$). If $v_i \neq 0$ for some i, then the system (5) is only solved by $x_1 = 0$.

2.2 Two vectors

Suppose we have a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, and suppose that this set is linearly dependent. Then there exist x_1, x_2 (not both zero) such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0},$$

or equivalently

$$x_1\mathbf{v}_1 = -x_2\mathbf{v}_2.$$

Thus, if a set of two vectors is linearly independent, then one is a multiple of the other.

Figure 1: A pair of sets of two vectors in $\mathbb{R}^2$. The set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent; indeed, $\mathbf{v}_2 = 1.5\mathbf{v}_1$.

What about the converse–is a set $\{\mathbf{v}_1, \mathbf{v}_2\}$, where one vector is a multiple of the other, linearly dependent? It is not hard to see that the answer is yes. Thus, we get the following characterization of linear dependence:

- A set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

3 More than two vectors

What about bigger sets of vectors, like $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$? Something like what we just proved for the case of two vectors is true, but just slightly more complicated.

Theorem 3.1. *The set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.*

Note that not every vector is a linear combination of the others! All that the theorem guarantees is the existence of some $\mathbf{v}_j$ which can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$, $\mathbf{v}_{j+1}, \ldots, \mathbf{v}_p$. Lay proves Theorem 3.1 in the section, and I highly recommend you read the proof. It will help you develop a sense of what linear dependence actually means, which will help you work the problems.

When we consider subsets of $\mathbb{R}^3$, there are some nice pictures that are associated with linear independence and dependence. One is the following: say we have three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$, and suppose the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent? If it is linearly dependent, then it is possible to write

$$\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some c_1, c_2 (why?) . That means that

$$\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \tag{6}$$

But since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane (see Lay 1.3, the subsection called "A geometric description of span $\{\mathbf{v}\}$ and span $\{\mathbf{u}, \mathbf{v}\}$"). Thus, (6) implies that $\mathbf{v}_3$ is in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

There are two nice theorems that let us tell that certain sets of vectors are linearly dependent without doing too much work. For both theorems, we will not "prove" them (the proofs are in Lay), but give concrete examples which give the idea of the proof. You can think about how to generalize these into a true proof or read the proofs in Lay.

Theorem 3.2. *If a set of vectors S contains the zero vector, then S is linearly dependent.*

Example 3.3. Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \, \mathbf{v}_2 = \mathbf{0}.$$

Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent?

Recall that linear dependence is equivalent to there existing weights x_1, x_2 which are not both equal to zero such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$. We can take $x_1 = 0$ and $x_2 = 1$:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0} = 0\mathbf{v}_1 + 1\mathbf{v_2} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So the set is linearly dependent, just as we said in Theorem 3.2. This sort of strategy works with any set of vectors containing $\mathbf{0}$, which proves the theorem.

The second theorem:

Theorem 3.4. *If a set contains more vectors than there are components in each vector, then the set is linearly dependent. I.e., if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ with $p > n$, then S is linearly dependent.*

Again, rather than "proving" in full detail, we will sketch the idea with an example.

Example 3.5. Say we have vectors

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ linearly dependent?

Write the augmented matrix

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}.$$

Since there are only two rows, there will be at most two pivot columns. However, there are three variables x_1, x_2, x_3. Thus, the variable x_3 will be free, so there will be a nontrivial solution. In case you don't believe me, you can check that the RREF is

$$\begin{bmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 1/2 & 0 \end{bmatrix},$$

so $-\mathbf{a}_1 - 2\mathbf{a}_2 + 4\mathbf{a}_3 = \mathbf{0}$.

A Appendix: Why do we care about linear dependence?

In this section, we try to describe exactly why you would care about linear dependence and independence. If you feel like the definitions make sense to you and you see why they would be useful, then this section is not so necessary for you.

Say we have a set of three vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Suppose though that $\mathbf{w}_1 + 4\mathbf{w}_2 = \mathbf{w}_3$, so you can write the vector $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Remember that span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is the set of all vectors that can be written as a linear combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. So suppose $\mathbf{y}$ is a vector in span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$:

$$\mathbf{y} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3.$$

Remember though that $\mathbf{w}_1 + 4\mathbf{w}_2 = \mathbf{w}_3$, so we can rewrite the above as

$$\begin{aligned}\mathbf{y} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3(\mathbf{w}_1 + 4\mathbf{w}_2)\\ &= (c_1 + c_3)\mathbf{w}_1 + (c_2 + 4c_3)\mathbf{w}_2,\end{aligned}$$

So

$$\mathbf{y} \in \text{span}\,\{\mathbf{w}_1, \mathbf{w}_2\}. \tag{7}$$

On the other hand, suppose we have some arbitrary vector $\mathbf{z} \in \text{span}\,\{\mathbf{w}_1, \mathbf{w}_2\}$. Then $\mathbf{z} \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ automatically, because adding vectors to a set only makes the span larger. What this and the reasoning leading up to (7) imply is that

$$\text{span}\,\{\mathbf{w}_1, \mathbf{w}_2\} = \text{span}\,\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}.$$

Let's summarize what we have learned here. If we have a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ which is linearly dependent, then we can remove vectors from this set without changing the span. This is a good reason to care about linear dependence. Later in the course, we will see that facts like the ones we discussed in this appendix have some very useful consequences.